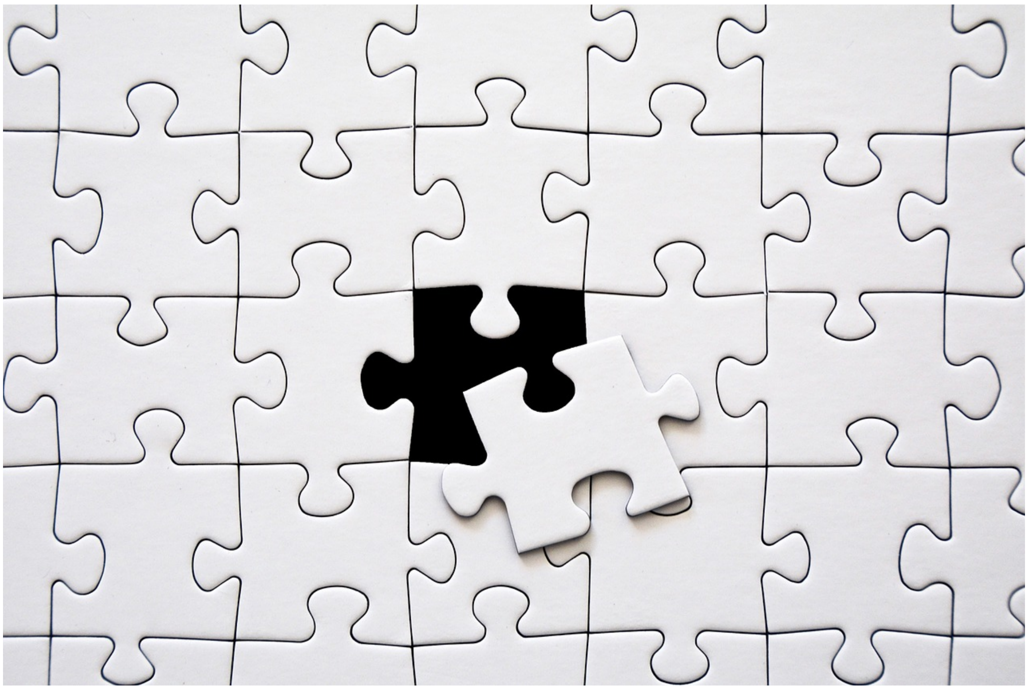

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Abstract

The integration of customer behavioral models in optimization provides a better understanding of the preferences of clients (the demand) to operators while planning for their systems (the supply). These preferences are formalized with discrete choice models, which are the state-of-the-art for the mathematical modeling of demand. However, their complexity leads to mathematical formulations that are highly nonlinear and nonconvex in the variables of interest, and are therefore difficult to be included in (mixed) integer linear problems (MILP). These problems correspond to the optimization models that are considered to design and configure a system. In this work, we present a general framework that integrates advanced discrete choice models in MILP. Nevertheless, a linear formulation comes with a high dimension of the problem. To address this issue, and given the underlying structure of the model, decomposition techniques such as Lagrangian decomposition can be applied. Two subproblems with common variables have been identified: one regarding the user and one regarding the operator. In the former, the user has to perform a decision based on what the operator is offering, whereas in the latter, the operator needs to decide about the features of the supply to attract the users.

Keywords

discrete choice models, combinatorial optimization, simulation, decomposition techniques

1 Introduction

Even though demand and supply closely interact in many applications, such as airlines or high speed railway, these two research fields have evolved independently, without paying too much attention to the existing interdependencies between the two. Indeed, incorporating the preferences and tastes of customers, which are usually characterized with discrete choice models, allows for a better planning of the systems for the operators. The design and configuration of such systems are typically addressed with optimization models, being MILP a relevant portion of the models reported in the literature.

In Pacheco *et al.* (2016), we propose a general framework that allows to integrate in MILP discrete choice models for which it is possible to draw from its associated probability distribution. This feature is important because we rely on simulation to circumvent the nonlinearity introduced by the demand model. To illustrate how the framework can be employed, an application on revenue maximization is considered. The performed experiments exhibit that the resulting formulation is a powerful tool to plan systems based on the heterogeneous behavior of customers. However, the disaggregate representation of the clients' behavior, together with the linearity of the formulation, leads to a computationally expensive problem (see Section 3.1).

To address this issue, we can benefit from the structure of the problem and explore decomposition techniques to solve it in an efficient way. These techniques are used in optimization problems that have the appropriate structure. They allow to speed up and facilitate the solution approach and/or to obtain good bounds for the optimal value of the objective function (Conejo *et al.*, 2006).

In practice, there are two different decomposable structures: one characterized by complicating constraints, with Lagrangian relaxation as the main associated decomposition technique, and another one characterized by complicating variables, with Benders decomposition as the principal technique. In our case, even if both could be applied, we focus on the former, since some constraints in the MILP can be seen as complicating because they are involved in both the demand and the supply model.

The remainder of the paper is organized as follows. In Section 2, the modeling framework is summarized, and the mathematical formulation for the application on revenue maximization is provided. Section 3 motivates the use of Lagrangian relaxation and characterizes the different problems comprised within this technique. Finally, in Section 4 we describe the future avenues of research associated with this investigation.

2 Modeling framework

In this approach, we embed a discrete choice model, which models the choices of customers, inside a MILP, which models the decisions of the operator. In this section, a summary of the mathematical formulation is provided.

2.1 General context

Regarding the demand model, we denote by C the choice set, which contains all potential services plus an *opt-out* option to capture the customers leaving the market (i.e., customers choosing a service from a competitor or not choosing anything at all). We consider a population of N customers, and we assume that the choice set of each customer may be different (e.g., due to different tastes, restrictions, etc.). The choice set of customer n is denoted by C_n.

The common specification of the utility associated with service i by customer n is

$$U_{in} = V_{in} + \varepsilon_{in}, \tag{1}$$

where V_{in} is the deterministic part and ε_{in} the error term, which is assumed to follow a probability distribution. We note that for the integration of the choice model in a MILP, the endogenous variables (present in both the demand and the supply models) need to appear linearly in the utility function. Thus, we assume V_{in} has the form

$$V_{in} = \sum_k \beta_{ink} x^e_{ink} + q^d(x^d), \tag{2}$$

where x^e_{ink} are the endogenous variables for service i and customer n (indexed by k), and $q^d(x^d)$ is a term that depends on other exogenous demand variables x^d (only present in the demand model), in a possibly nonlinear way defined by the function q^d.

The behavioral assumption is that customer n chooses service i if the associated utility is the largest within the choice set C_n. The probability that customer n chooses service i is

$$P_n(i|C_n) = \Pr(U_{in} \geq U_{jn}, \forall j \in C_n). \tag{3}$$

This formulation is typically nonlinear as a function of the endogenous variables. For the sake of the integration, we work directly with the utility functions, and not with the corresponding probabilities. To deal with the random nature of the utility function, we rely on simulation.

For each error term ε_{in}, we generate R draws $\xi_{in1},\ldots,\xi_{inR}$ based on its distributional assumption. Each draw corresponds to a behavioral scenario. For the specification (1), we have

$$U_{inr} = V_{in} + \xi_{inr} = \sum_k \beta_{ink} x^e_{ink} + q^d(x^d) + \xi_{inr}. \tag{4}$$

A relevant application to illustrate the described methodology is the maximization of revenue. We use this example to characterize a concrete MILP. The objective is to find the best strategy in terms of pricing and capacity allocation in order to maximize the revenue of the operator, which sells services at a certain price (to be decided) and with a given capacity.

In this formulation, the price is the only endogenous variable, appearing in both the utility function (demand) and the objective function (supply). We define $p_{in} \in \mathbb{R}$ as the price that customer n must pay to access service i. We assume it can take a finite number of values, called *price levels*, between a lower bound ℓ_{in} and an upper bound m_{in} (bounds on the integer representation of the price). For linearization issues, it is characterized as follows:

$$p_{in} = \frac{1}{10^k}\Big(\ell_{in} + \sum_{\ell=0}^{L_{in}-1} 2^\ell \lambda_{in\ell}\Big), \tag{5}$$

where the term in brackets corresponds to the binary representation of the price as a integer value ($\lambda_{in\ell}$ are the associated binary variables, and there are L_{in} of such variables), and $\frac{1}{10^k}$ transforms the integer value into the actual continuous value of the price (with a precision of k decimals). The decision variables are therefore $\lambda_{in\ell}$, and not p_{in}.

2.2 Mathematical formulation

The mathematical formulation for the application described above is depicted in Fig. 1. We provide a brief description of its sets, input parameters, variables, objective function and constraints.

Sets

- $I = |C|$: number of services, indexed by i ($i = 0$ denotes the opt-out option)
- L_{in}: number of binary variables characterizing the price levels, indexed by ℓ
- N: number of customers, indexed by n
- R: number of draws, indexed by r

Input parameters

- c_i: capacity of service i
- ℓ_{in}, m_{in}: integer bounds on the price (set by the operator)
- $\ell_{inr} \leq U_{inr} \leq m_{inr}$: bounds on U_{inr} (p_{in} is bounded and $q^d(x^d)$ is given)
- $\ell_{nr} = \min_{j \in C_n} \ell_{jnr}$: smallest lower bound for customer n and scenario r
- $m_{nr} = \max_{j \in C_n} m_{jnr}$: largest upper bound for customer n and scenario r
- $M_{inr} = m_{inr} - \ell_{nr}$, $M_{nr} = m_{nr} - \ell_{nr}$ (big-M constraints)
- $q^d(x^d)$: term depending on exogenous demand variables (given)
- β_{in}: parameters associated with the price variables p_{in} (for the sake of compactness of the formulation, we impose $\beta_{in} = 0$ for $i = 0$, so that the price term vanishes)
- ξ_{inr}: r-th draw from the error term for service i and customer n

Variables

- U_{nr}: maximum discounted utility, $U_{nr} = \max_{i \in C} z_{inr}$
- $y_{in} \in \{0, 1\}$: availability at operator level, 1 if service i is available to customer n and 0 otherwise (explicit decision of the operator)
- $y_{inr} \in \{0, 1\}$: availability at scenario level, 1 if service i is available to customer n in scenario r and 0 otherwise (result of the choices of other customers when service capacity is insufficient to satisfy the total demand)
- z_{inr}: discounted utility, U_{inr} when the service is available and ℓ_{nr} otherwise
- $w_{inr} \in \{0, 1\}$: choice, 1 if service i is chosen by customer n in scenario r and 0 otherwise
- $\alpha_{inr\ell} \in \{0, 1\}$: linearization of the product $w_{inr}\lambda_{in\ell}$, 1 if $w_{inr}\lambda_{in\ell} = 1$ and 0 otherwise
- $\lambda_{in\ell} \in \{0, 1\}$: binary representation of the price (see (5))

Objective function Maximization of the total expected revenue (6), which is obtained by adding the revenues from all services but the opt-out. The revenue of each service is calculated with the associated price and the expected demand (obtained from the choice variables).

Constraints

- Utility: (7) defines the utility associated by customer n with service i in the r-th scenario (obtained from (4) with the price as the only endogenous variable)
- Availability: (8) makes service i unavailable if $i \notin C_n$ and (9) relates both availabilities (a service cannot be available at scenario level if it is not made available by the operator)
- Discounted utility: (10)–(13) correspond to the linear formulation of z_{inr}

- Choice: (14)–(17) characterize the choice (i.e., service i is chosen by customer n in scenario r if i is the argument of the maximum discounted utility)
- Capacity allocation: (18)–(20) handle the capacity limitations
- Pricing: (21) bounds the price from above and (22)–(24) are the linearizing constraints associated with $\alpha_{inr\ell}$

Figure 1: MILP for the demand-base revenue maximization application

$$\max \frac{1}{R}\frac{1}{10^k}\Big[\sum_n \sum_{i>0, i\in C_n} \sum_r \Big(\ell_{in} w_{inr} + \sum_\ell 2^\ell \alpha_{inr\ell}\Big)\Big] \tag{6}$$

subject to

$$U_{inr} = \beta_{in}\frac{1}{10^k}\Big(\ell_{in} + \sum_\ell 2^\ell \lambda_{in\ell}\Big) + q_d(x_d) + \xi_{inr}, \qquad \forall i \in C_n, n, r, \tag{7}$$

$$y_{in} = 0, \qquad \forall i \notin C_n, n, r, \tag{8}$$

$$y_{inr} \leq y_{in}, \qquad \forall i, n, r, \tag{9}$$

$$\ell_{nr} \leq z_{inr}, \qquad \forall i, n, r, \tag{10}$$

$$z_{inr} \leq \ell_{nr} + M_{inr} y_{inr}, \qquad \forall i, n, r, \tag{11}$$

$$U_{inr} - M_{inr}(1 - y_{inr}) \leq z_{inr}, \qquad \forall i, n, r, \tag{12}$$

$$z_{inr} \leq U_{inr}, \qquad \forall i, n, r, \tag{13}$$

$$\sum_i w_{inr} = 1, \qquad \forall n, r, \tag{14}$$

$$w_{inr} \leq y_{inr}, \qquad \forall i, n, r, \tag{15}$$

$$z_{inr} \leq U_{nr}, \qquad \forall i, n, r, \tag{16}$$

$$U_{nr} \leq z_{inr} + M_{nr}(1 - w_{inr}), \qquad \forall i, n, r, \tag{17}$$

$$y_{inr} \geq y_{i(n+1)r}, \qquad \forall i > 0, n < N, r, \tag{18}$$

$$\sum_{m=1}^{n-1} w_{imr} \leq (c_i - 1) y_{inr} + (n-1)(1 - y_{inr}), \qquad \forall i > 0, i \in C_n, n > c_i, r, \tag{19}$$

$$c_i(y_{in} - y_{inr}) \leq \sum_{m=1}^{n-1} w_{imr}, \qquad \forall i > 0, n, r, \tag{20}$$

$$\ell_{in} + \sum_\ell 2^\ell \lambda_{in\ell} \leq m_{in}, \qquad \forall i > 0, i \in C_n, n, \tag{21}$$

$$\lambda_{in\ell} + w_{inr} \leq 1 + \alpha_{inr\ell}, \qquad \forall i > 0, i \in C_n, n, r, \ell, \tag{22}$$

$$\alpha_{inr\ell} \leq \lambda_{in\ell}, \qquad \forall i > 0, i \in C_n, n, r, \ell, \tag{23}$$

$$\alpha_{inr\ell} \leq w_{inr}, \qquad \forall i > 0, i \in C_n, n, r, \ell. \tag{24}$$

3 A Lagrangian relaxation method for the uncapacitated case

3.1 Motivation

The framework described in Section 2 has been tested on a case study from the recent literature. We have considered a parking services operator, whose disaggregate choice model is characterized in Ibeas *et al.* (2014). More precisely, they estimate a mixtures of a logit model (i.e., allowing for different coefficients among customers) to describe the behavior of potential car park users when choosing a parking place.

For the experiments performed in Pacheco *et al.* (forthcoming), we consider a sample of $N = 50$ customers to avoid solving huge optimization problems. The choice set is composed of $I = 3$ services (including the opt-out option), and $L = 4$ binary variables are used to characterize the price, giving rise to 16 possible price levels ($L_{in} = L, \forall i, n$). Then, different values of R have been employed to evaluate the complexity. For $R = 250$ (the largest number of draws considered), the uncapacitated case (when services are assumed to have unlimited capacity, i.e., constraints (18)–(20) are ignored) takes 2.5 hours, whereas the capacitated case (the full model in Figure 1) takes almost 42 hours.

These results confirm that the problem is computationally expensive, even for instances of moderate size. In practice, populations are way larger and a high number of draws is desirable to be as close as possible to the true value, so it is important to develop methodologies in order to overcome this limitation. As mentioned in Section 1, we consider Lagrangian relaxation to speed up the solution approach.

In this technique, the constraints that are considered hard are relaxed by bringing them to the objective function with associated parameters, called *Lagrangian multipliers*. The resulting optimization problem is called *Lagrangian subproblem*. It holds that the optimal value of the Lagrangian subproblem is a lower (upper) bound on the optimal value of the original minimization (maximization) problem. In order to obtain the tightest possible lower (upper) bound, an optimization problem on the Lagrangian multipliers, called the *Lagrangian dual* problem associated with the original optimization problem, is solved.

The remainder of this section is organized as follows. In Section 3.2, we define a simpler optimization problem to start with by making some assumptions on the problem modeled in Figure 1, and we characterize the corresponding Lagrangian subproblem (Figure 3). We note that it can be split into two subproblems with common variables: the *choice subproblem*, concerning the choice made by each customer (Figure 4), and the *price subproblem*, concerning the selection of the price level for each service (Figure 6). We provide an algorithm to solve each subproblem (Figure 5 and Figure 8, respectively). In Section 3.3, we detail the associated Lagrangian dual (63) and we approximate it with a subgradient method (Figure 9).

3.2 Lagrangian subproblem

Given the complexity introduced by the capacity constraints (18)–(20), we consider at first the uncapacitated case to characterize the Lagrangian subproblem. Indeed, as soon as we forget about capacity, the problem is reduced to that of assigning the service with the highest utility to each customer (among the available ones) and of computing the price for each service. Other considerations that are taken into account are itemized next. The resulting formulation is presented in Figure 2.

Availability Since the capacity of the services is unlimited, the variables for the availability at scenario level (y_{inr}) are not needed (they are all equal to 1). Furthermore, the availability at operator level is characterized by means of the subsets C_n, and not with the variables y_{in}. This is a simplification with respect to the model in Figure 1, since it does not enable the operator to close and open services depending on the arriving customers. However, it can fit some contexts where it is not possible to decide on the availability of the services in an online way (e.g., parking services). Consequently, we get rid of constraints (8)–(9) and (15).

Discounted utility Given that there is no availability at scenario level, the concept of discounted utility is not necessary, and the value of the utility can be used directly (i.e., $z_{inr} = U_{inr}, \forall i \in C_n, n, r$). Then, constraints (10)–(11) involve the bounds on the utility

$$l_{inr} \leq U_{inr} \leq m_{inr}, \tag{25}$$

and constraints (12)–(13) are redundant, and can therefore be ignored.

Choice In this approach, we remove the variables U_{nr} and we define the choice variable as follows:

$$w_{inr} = \begin{cases} 1 & \text{if } i = \arg\max_{n, i \in C_n}\{U_{inr}\} \\ 0 & \text{otherwise,} \end{cases} \quad \forall i, n, r. \tag{26}$$

With this specification the model becomes nonlinear (we refer to it as *pseudo-MILP*).

Figure 2: Pseudo-MILP for the uncapacitated case

$$\max \quad \frac{1}{R}\frac{1}{10^k}\sum_{i>0, i\in C_n}\Big[\sum_n\sum_r\Big(\ell_{in}w_{inr}+\sum_\ell 2^\ell\alpha_{inr\ell}\Big)\Big] \tag{27}$$

s.t.

$$U_{inr}=\beta_{in}\frac{1}{10^k}\sum_\ell 2^\ell\lambda_{in\ell}+b_{inr}, \qquad \forall i\in C_n, n, r, \tag{28}$$

$$\ell_{inr}\leq U_{inr}\leq m_{inr}, \qquad \forall i\in C_n, n, r, \tag{29}$$

$$w_{inr}=\begin{cases}1 & \text{if } i=\arg\max_{n,i\in C_n}\{U_{inr}\},\\ 0 & \text{otherwise,}\end{cases} \qquad \forall i, n, r, \tag{30}$$

$$\sum_i w_{inr}=1, \qquad \forall n, r, \tag{31}$$

$$\ell_{in}+\sum_\ell 2^\ell\lambda_{in\ell}\leq m_{in}, \qquad \forall i>0, i\in C_n, n, \tag{32}$$

$$\alpha_{inr\ell}\leq\lambda_{in\ell}, \qquad \forall i>0, i\in C_n, n, r, \ell, \tag{33}$$

$$\alpha_{inr\ell}\leq w_{inr}, \qquad \forall i>0, i\in C_n, n, r, \ell, \tag{34}$$

$$\text{where} \quad b_{inr}=\beta_{in}\frac{1}{10^k}\ell_{in}+q_d(x_d)+\xi_{inr} \qquad \forall i\in C_n, n, r. \tag{35}$$

Capacity We ignore constraints (18)–(20) to account for unlimited capacity of services.

Pricing Constraint (22) is not needed because of constraints (23)–(24) and the objective function (6). This can be verified by considering two cases:

- $\lambda_{inl}=1\wedge w_{inr}=1\Rightarrow\alpha_{inr}=1$: constraints (23)–(24) are always verified, but do not force α_{inr} to be equal to 1. However, since the objective function is to be maximized and these variables appear positively in the objective function, they are pushed to the upper bound, that is 1, so that constraint (22) can be ignored.
- $\lambda_{inl}=0\vee w_{inr}=0\Rightarrow\alpha_{inr}=0$: constraints (23)–(24) already force $\alpha_{inr}=0$ (regardless of the objective function), and therefore constraint (22) can also be ignored.

We note that constraint (22) could have also been removed in Figure 1, but as mentioned in Section 2.1, we are aiming at keeping a general framework, and the MILP in Figure 1 is just a concrete characterization (i.e., with a particular objective function).

The constraints associated with the choice performed by the customers are (28)–(31), whereas the ones associated with the price selection are (32)–(34). We note that these sets of constraints have a common variable: w_{inr}. In order to separate the Lagrangian subproblem into the choice and the price subproblems, we introduce copy variables v_{inr} and the corresponding copy constraints (which are relaxed in a Lagrangian way):

$$v_{inr} = w_{inr}, \qquad \forall i, n, r. \tag{36}$$

We take advantage of the structure of the problem and we write constraint (36) as follows:

$$v_{inr} \leq w_{inr}, \qquad \forall i, n, r, \tag{37}$$

$$\sum_i v_{inr} = 1, \qquad \forall n, r. \tag{38}$$

To prove the equivalence between (36) and (37)–(38), assume that for a certain customer n and scenario r, there exists a service j such that $v_{jnr} > w_{jnr}$. This implies $v_{jnr} = 1$ (since $\sum_i v_{inr} = 1$) and $w_{jnr} = 0$. So there must exist j' such that $w_{j'nr} = 1$ (since $\sum_i w_{inr} = 1$) and $v_{j'nr} = 0$ (since $v_{jnr} = 1$). But this implies $v_{j'nr} < w_{j'nr}$, which is a contradiction to our original assumption. Hence, we necessarily have $v_{inr} = w_{inr}$ $\forall i, n, r$.

The advantage of constraints (37)–(38) is twofold. On the one hand, the replacement of the equality in (36) by the inequality in (37) introduces non-negative Lagrange multipliers (instead of unconstrained multipliers), which facilitates the corresponding optimization. On the other hand, we introduce redundant assignment constraints (38), which strengths the Lagrangian subproblem. Constraint (37) is the one relaxed in a Lagrangian way.

In addition to constraint (37), constraint (28) defining the utility variable U_{inr} is also transferred to the objective function with unconstrained Lagrange multipliers. This allows us to decouple the choice subproblem from the price subproblem. The resulting Lagrangian subproblem is depicted in Figure 3. We denote by $\theta_{inr} \in \mathbb{R}$ the Lagrangian multipliers associated with constraint (28), and by $\gamma_{inr} \geq 0$ the ones associated with constraint (37).

The choice subproblem involves the variables U_{inr} and w_{inr}, and comprises constraints (40)–(42) (Figure 4), whereas the price subproblem involves λ_{inl}, α_{inrl} and v_{inr}, and comprises constraints (43)–(46) (Figure 6). In both cases, the objective function is composed by the terms depending on the corresponding variables.

Furthermore, two decomposition sources are identified in the model in Figure 1. On the one hand, each draw r represents an independent scenario, and all scenarios are coupled only in the objective function (6). On the other hand, each customer n aims at choosing the service among the available ones maximizing her utility, and all customers are coupled in the capacity constraints (18)–(20), to ensure that the capacity of each alternative is not exceeded, and in the objective function (6), to calculate the total demand. For the Lagrangian subproblem defined in Figure 3, the choice subproblem decomposes by n and r (i.e., a subproblem is solved for each customer and draw), and the price subproblem decomposes by n (i.e., a subproblem is solved for each customer). We denote the subproblems by $Z^c_{nr}(\theta,\gamma)$ and $Z^p_n(\theta,\gamma)$, respectively.

Figure 3: Lagrangian subproblem

$$Z(\theta,\gamma) = \max \frac{1}{R}\frac{1}{10^k}\Big[\sum_n \sum_{i>0,i\in C_n} \sum_r \Big(\ell_{in} w_{inr} + \sum_\ell 2^\ell \alpha_{inr\ell}\Big)\Big] \tag{39}$$
$$+ \sum_n \sum_{i\in C_n} \sum_r \theta_{inr}\Big(U_{inr} - \beta_{in}\frac{1}{10^k}\sum_\ell 2^\ell \lambda_{in\ell} - b_{inr}\Big)$$
$$+ \sum_i \sum_n \sum_r \gamma_{inr}(v_{inr} - w_{inr})$$

s.t.

$$\ell_{inr} \leq U_{inr} \leq m_{inr} \qquad \forall i>0, i\in C_n, n, r \tag{40}$$
$$w_{inr} = \begin{cases} 1 & \text{if } i = \arg\max_{n,i\in C_n}\{U_{inr}\}, \\ 0 & \text{otherwise,} \end{cases} \qquad \forall i, n, r, \tag{41}$$
$$\sum_i w_{inr} = 1, \qquad \forall n, r, \tag{42}$$
$$\ell_{in} + \sum_\ell 2^\ell \lambda_{in\ell} \leq m_{in}, \qquad \forall i>0, i\in C_n, n, \tag{43}$$
$$\alpha_{inr\ell} \leq \lambda_{in\ell}, \qquad \forall i>0, i\in C_n, n, r, \ell, \tag{44}$$
$$\alpha_{inr\ell} \leq v_{inr}, \qquad \forall i>0, i\in C_n, n, r, \ell, \tag{45}$$
$$\sum_i v_{inr} = 1, \qquad \forall n, r. \tag{46}$$

3.2.1 Choice subproblem

The choice subproblem for each customer n and draw r is defined in Figure 4. The solution of this problem is easy to compute by taking into account the following observations:

- The value of U_{inr} is either m_{inr} or ℓ_{inr}. Indeed, if $\theta_{inr} > 0$, then $U_{inr} = m_{inr}$, since the objective function is to be maximized and the term $\theta_{inr}U_{inr}$ contributes positively. If $\theta_{inr} < 0$, then $U_{inr} = \ell_{inr}$, since the term $\theta_{inr}U_{inr}$ contributes negatively. If $\theta_{inr} = 0$, U_{inr} can take any value, so for the sake of simplicity we merge this case within $\theta_{inr} > 0$.

- Constraint (49) states that the service within C_n with the highest value of U_{inr} (i.e., U_{nr}) is chosen, and constraint (50) implies that only one service can be selected. Thus, the service $i \in C_n$ such that $\max_{i|\theta_{inr}\geq 0}\{m_{inr}\}$ and $\max_{i|\theta_{inr}<0}\{\ell_{inr}\}$ is the one chosen.
- If several $i \in C_n$ achieve U_{nr}, we must choose the one with the highest contribution to the objective function, i.e., with the largest $\{\ell_{in} + \theta_{inr}U_{nr} - \gamma_{inr}\}$.

Figure 4: Choice subproblem

$$Z^c_{nr}(\theta,\gamma) = \max \frac{1}{R}\frac{1}{10^k}\sum_{i>0,i\in C_n} \ell_{in}w_{inr} + \sum_{i\in C_n}\theta_{inr}U_{inr} - \sum_i \gamma_{inr}w_{inr} \tag{47}$$

s.t.

$$\ell_{inr} \leq U_{inr} \leq m_{inr}, \qquad \forall i \in C_n, \tag{48}$$

$$w_{inr} = \begin{cases} 1 & \text{if } i = \arg\max_{n,i\in C_n}\{U_{inr}\}, \\ 0 & \text{otherwise,} \end{cases} \qquad \forall i, \tag{49}$$

$$\sum_i w_{inr} = 1. \tag{50}$$

The algorithm to compute $Z^C_{nr}(\theta,\gamma)$, based on the above observations, is described in Figure 5.

Figure 5: Algorithm to solve the choice subproblem

Input: $\theta_{inr}, \gamma_{inr}$

Set $U_{inr} = \begin{cases} m_{inr} & \text{if } \theta_{inr} \geq 0 \\ \ell_{inr} & \text{otherwise} \end{cases} \quad \forall i \in C_n$

Obtain i where $U_{nr} = \max_{i\in C_n} U_{inr}$ is achieved

Compute $i^* = \arg\max_{i\in C_n}\{\ell_{in} + \theta_{inr}U_{nr} - \gamma_{inr}\}$

Output: $Z^c_{nr}(\theta,\gamma) = \begin{cases} \frac{1}{R}\frac{1}{10^k}\ell_{i^*n} + \sum_{i\in C_n}\theta_{inr}U_{inr} - \gamma_{i^*nr}, & \text{if } i^* \neq 0, \\ \sum_{i\in C_n}\theta_{inr}U_{inr} - \gamma_{i^*nr}, & \text{if } i^* = 0. \end{cases}$

3.2.2 Price subproblem

The price subproblem for each customer n is defined in Figure 6. For each draw r, the general idea is to calculate the contribution to the objective function for each service i when it is chosen ($v_{inr} = 1$) and when it is not ($v_{inr} = 0$). Then, the service with the highest contribution for that draw is chosen, and the associated price is computed.

Figure 6: Price subproblem

$$Z_n^p(\theta,\gamma) = \max \quad \frac{1}{R}\frac{1}{10^k}\sum_{i>0,i\in C_n}\sum_r\sum_\ell 2^\ell \alpha_{inr\ell} - \frac{1}{10^k}\sum_{i\in C_n}\sum_r \theta_{inr}\beta_{in}\sum_\ell 2^\ell\lambda_{in\ell} + \sum_i \gamma_{inr}v_{inr} \tag{51}$$

s.t.

$$\ell_{in} + \sum_\ell 2^\ell \lambda_{in\ell} \leq m_{in}, \quad \forall i>0, i\in C_n, \tag{52}$$

$$\alpha_{inr\ell} \leq \lambda_{in\ell}, \quad \forall i>0, i\in C_n, r, \ell, \tag{53}$$

$$\alpha_{inr\ell} \leq v_{inr}, \quad \forall i>0, i\in C_n, r, \ell, \tag{54}$$

$$\sum_i v_{inr} = 1, \quad \forall r. \tag{55}$$

Unchosen service If $v_{inr} = 0$, then $\alpha_{inr\ell} = 0\ \forall \ell$ (due to constraint (54)). The contribution to the objective function for service i and draw r is

$$\zeta^0_{inr}(\theta,\gamma,\lambda) = -\frac{1}{10^k}\theta_{inr}\beta_{in}\sum_\ell 2^\ell\lambda_{in\ell}. \tag{56}$$

We can consider three cases to characterize ζ^0_{inr} based on the values of $\theta_{inr}\beta_{in}$:

1. If $\theta_{inr}\beta_{in} < 0$, then ζ^0_{inr} is always positive, and since the objective function is to be maximized, we obtain $\lambda_{in\ell} = 1\ \forall\ell$ such that $\sum_\ell 2^\ell\lambda_{in\ell} = m_{in} - \ell_{in}$ (the remaining $\lambda_{in\ell}$ are equal to 0).
2. If $\theta_{inr}\beta_{in} > 0$, then $\lambda_{in\ell} = 0\ \forall\ell$, since ζ^0_{inr} is always negative and we are maximizing the objective function.
3. If $\theta_{inr}\beta_{in} = 0$, then $\zeta^0_{inr} = 0$, so $\lambda_{in\ell}$ can take any value. We merge this case within case 2, so that $\lambda_{in\ell} = 0\ \forall\ell$.

We note that for $i = 0$, $\zeta^0_{inr}(\theta,\gamma,\lambda)$ is not defined, since the price variables are only defined for $i > 0$. However, as we have set $\beta_{in} = 0$ for $i = 0$, we can consider a contribution of 0 for $i = 0$, and merge this possibility under case 3. By taking all possibilities into account, we define $\zeta^0_{inr}(\theta,\gamma)$ as follows:

$$\zeta^0_{inr}(\theta,\gamma) = \begin{cases} -\frac{1}{10^k}\theta_{inr}\beta_{in}(m_{in}-\ell_{in}), & \text{if } \theta_{inr}\beta_{in} < 0, \\ 0, & \text{if } \theta_{inr}\beta_{in} \geq 0 \lor i = 0, \end{cases} \quad \forall i \in C_n, r. \tag{57}$$

Chosen service If $v_{inr} = 1$, then $\alpha_{inr\ell} = \lambda_{inl}\ \forall\ell$ (due to constraint (53)). The contribution to the objective function for service i and draw r is

$$\zeta^1_{inr}(\theta,\gamma,\lambda) = \frac{1}{10^k}\left(\frac{1}{R} - \theta_{inr}\beta_{in}\right)\sum_\ell 2^\ell \lambda_{in\ell} + \gamma_{inr}. \tag{58}$$

We can consider three cases to characterize ζ^1_{inr} based on the values of $\left\{\frac{1}{R} - \theta_{inr}\beta_{in}\right\}$:

1. If $\frac{1}{R} - \theta_{inr}\beta_{in} > 0$, then ζ^1_{inr} is always positive ($\gamma_{inr} \geq 0$), and since the objective function is to be maximized, we obtain $\lambda_{in\ell} = 1\ \forall\ell$ such that $\sum_\ell 2^\ell\lambda_{in\ell} = m_{in} - \ell_{in}$.
2. If $\frac{1}{R} - \theta_{inr}\beta_{in} < 0$, then $\lambda_{in\ell} = 0\ \forall\ell$ so that ζ^1_{inr} takes the highest possible value (i.e., γ_{inr}).
3. If $\frac{1}{R} - \theta_{inr}\beta_{in} = 0$, then $\zeta^1_{inr} = \gamma_{inr}$ and $\lambda_{in\ell}$ can take any value. We merge this case within case 2, so that $\lambda_{in\ell} = 0\ \forall\ell$.

We note that for $i = 0$, $\zeta^1_{inr}(\theta,\gamma) = \gamma_{inr}$ (the other term is not defined). By taking all possibilities into account, we define $\zeta^1_{inr}(\theta,\gamma)$ as follows:

$$\zeta^1_{inr}(\theta,\gamma) = \begin{cases} \frac{1}{10^k}\left(\frac{1}{R} - \theta_{inr}\beta_{in}\right)(m_{in} - \ell_{in}) + \gamma_{inr}, & \text{if } \frac{1}{R} - \theta_{inr}\beta_{in} > 0, \\ \gamma_{inr}, & \text{if } \frac{1}{R} - \theta_{inr}\beta_{in} \leq 0 \lor i = 0, \end{cases} \quad \forall i \in C_n, r. \tag{59}$$

We can write the price subproblem in Figure 6 in terms of ζ^0_{inr} and ζ^1_{inr}. The resulting formulation is defined in Figure 7. This problem is easy to solve. For each draw r, the chosen service is the one with the highest contribution to the objective function, i.e., $i^*_r = \arg\max \zeta^1_{inr}(\theta,\gamma)$. The complete algorithm is described in Figure 8.

Figure 7: Price subproblem in terms of ζ^0_{inr} and ζ^1_{inr}

$$Z^p_n(\theta,\gamma) = \max \quad \sum_{i\in C_n}\sum_r \zeta^0_{inr}(1 - v_{inr}) + \zeta^1_{inr}v_{inr} \tag{60}$$

s.t.

$$\sum_{i\in C_n} v_{inr} = 1, \qquad \forall r. \tag{61}$$

Figure 8: Algorithm for the price subproblem

Input: $\theta_{inr}, \gamma_{inr}\ \forall i \in C_n, r$
Compute ζ^0_{inr} and ζ^1_{inr}
Solve the problem in Figure 7 by computing $i^*_r = \arg\max_{i \in C_n} \zeta^1_{inr}\ \forall r$
Set $v_{inr} = \begin{cases} 1 & \text{if } i = i^*_r \\ 0 & \text{otherwise} \end{cases} \quad \forall i, r.$
Output: $Z^p_n(\theta, \gamma) = \sum_r \zeta^1_{i^*_r nr} + \sum_{i \neq i^*_r} \sum_r \zeta^0_{inr}$

3.3 Lagrangian dual

As mentioned in Section 3.1, the solution of the Lagrangian relaxation provides a lower (upper) bound of the initial minimization (maximization) problem. Furthermore, in the case of (convex) linear programs, the optimal solution of the Lagrangian subproblem coincides with the optimal solution of the initial problem.

In this case, the Lagrangian subproblem presented in Figure 3 is written as

$$Z(\theta, \gamma) = \sum_n \sum_r Z^c_{nr}(\theta, \gamma) + \sum_n Z^p_n(\theta, \gamma) - \sum_n \sum_{i \in C_n} \sum_r \theta_{inr} b_{inr}. \tag{62}$$

Since the original problem is a maximization problem, the Lagrangian dual is defined as

$$\min_{\theta, \gamma} Z(\theta, \gamma). \tag{63}$$

The Lagrangian dual can be approximated by an iterative method that updates the values of the Lagrangian multipliers by solving the so-called relaxed primal problem (RPP), which is the Lagrangian subproblem for the given values of the multipliers. The number of iterations depends on the desired accuracy of the result, and it can be set by the analyst or can obey a concrete stopping criterion.

There are different procedures for updating the Lagrangian multipliers. Here we consider the subgradient method, because it is simple to implement and its computational burden is small. Figure 9 shows the algorithm applied to our case. In step 1, we initialize the values of the Lagrangian multipliers. In step 2, the Lagrangian subproblem (62) is solved for the given values of the multipliers, obtaining values for the variables of the problem. The multipliers are updated in step 3 by calculating a subgradient of the function $Z(\theta, \gamma)$ at the multipliers of the current iteration (k). The subgradients are obtained as follows:

$$g^k_{inr} = U^k_{inr} - \beta_{in}\frac{1}{10^k}\sum_{\ell} 2^{\ell}\lambda^k_{in\ell} - b_{inr} \qquad \forall i \in C_n, n, r, \tag{64}$$

$$h^k_{inr} = v^k_{inr} - w^k_{inr} \qquad \forall i, n, r. \tag{65}$$

If they are equal to 0, the optimal solution is $Z(\theta^k, \gamma^k)$. If not, the multipliers are updated. To do so, the step size needs to be computed. Many different types of step size rules are used. For the sake of simplicity, we consider a constant step size (δ). As the multipliers associated with equality constraints are free, the update is not restricted (i.e., $\theta^{k+1}_{inr} = \theta^k_{inr} + \delta \cdot g^k_{inr}$, $\forall i \in C_n, n, r, k$), whereas for the multipliers associated with "less or equal" constraints, we have to ensure that they are positive. Thus, the multipliers take the maximum value between 0 and the value of the update (i.e., $\gamma^{k+1}_{inr} = \max\{0, \gamma^k_{inr} + \delta \cdot h^k_{inr}\}$, $\forall i, n, r, k$).

Figure 9: Subgradient method

1. Initialization: set $k = 0$ and choose θ^0_{inr} $\forall i \in C_n, n, r$ and γ^0_{inr} $\forall i, n, r$;
2. Solution of the RPP: solve $Z(\theta^k, \gamma^k)$ and obtain values for the variables U^k_{inr}, w^k_{inr}, $\lambda^k_{in\ell}$, $\alpha^k_{inr\ell}$ and v^k_{inr} where it is achieved;
3. Update the multipliers: choose subgradients g^k, h^k of the function $Z(\theta, \gamma)$ at θ^k and γ^k (g^k associated with the relaxation of the utility constraint and h^k associated with the copy constraint).
The subgradients g^k, h^k are obtained with (64)–(65) ;
if $(g^k, h^k)^T = 0$ **then**
    the optimal solution is $Z(\theta^k, \gamma^k)$, **stop**;
else
    compute $\theta^{k+1}_{inr} = \theta^k_{inr} + \delta \cdot g^k_{inr}$, $\forall i \in C_n, n, r$;
    compute $\gamma^{k+1}_{inr} = \max\{0, \gamma^k_{inr} + \delta \cdot h^k_{inr}\}$, $\forall i, n, r$;
    increment k and go to step 2;
end

The subgradient method provides an upper bound of the original problem Figure 2. In order to obtain a lower bound, we can follow these steps:

1. Fix the prices of the price subproblem.
2. Compute the choice of each individual based on the fixed prices.
3. Compute the original objective function value based on the prices from step 1 and the choices from step 2.

4 Conclusions and future work

The Lagrangian relaxation technique described in Section 3 has to be evaluated and compared with the results obtained with the exact method. We note that in the case study considered for the exact method, the services were assumed to be accessible to all customers at an operator level. Thus, both approaches can be safely compared, since the feature enabling the operator to open or close a service depending on the upcoming customer is deactivated.

Despite the simplicity of the described technique, the upper bound provided by the Lagrangian subproblem is restricted, in the sense that the price variables p_{in} and the utility variables U_{inr} can only take the associated extreme values. In practice, however, this is not the case and intermediate price levels and values of the utility are achieved. Hence, the proposed technique can be modified in order to define more complex subproblems, so that less importance is placed in the subgradient method. One possibility to be explored is to duplicate the price variables instead of the choice variables, since as soon as the former are fixed, the utiliy values can be easily computed, and so the choices performed by the customers.

The extension of the method to the capacitated case is simple. Indeed, we notice that only the choice subproblem has to be adapted, as the capacity constraints determine the set of available alternatives to choose from. Since the priority list determining the order in which customers are processed is assumed to be known, it is possible to iterate over the customers in the given order (the subproblem now only decomposes by r), and track the remaining capacity of each alternative. As soon as an alternative has reached its capacity, it is not offered anymore to the upcoming individuals. This is done by updating the associated individual choice sets C_n.

As a future avenue of research, in case the difference between the upper and lower bound for the original problem is significant, column generation can be used to define an exact method to solve the Lagrangian dual. The attractiveness of column generation is to work only with a sufficiently meaningful subset of variables, and to add more variables only when needed.

5 References

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